Sheaf cohomology and beginning the study of curves.

Recap of sheaf cohomology:

X scheme of finite type over

sheaf cohomology is a functor

$$QCoh(X) \longrightarrow Gr\,Vect_k \,/\, QCoh(X)$$

$$\mathcal{F} \longmapsto \left(H^i(X,\mathcal{F})\right)_{i\in\mathbb{Z}}.$$

such that

(1) $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$

(2) for every short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

we have a long exact sequence in cohomology

$$0 \to H^0(X,\mathcal{F}') \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}'') \to H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}'') \to \cdots$$

(3) For every morphism $\pi: X \to Y$ there is a natural homo

$$H^0(Y,\pi_*\mathcal{F}) \to H^*(X,\pi_*\mathcal{F})$$

which is an iso if π is affine.

(4) $H^i(X, \operatorname{colim} \mathcal{F}_j) = \operatorname{colim} H^i(X, \mathcal{F}_j)$

(5) if X is proper, $H^i(X, \mathcal{F})$ is fin. dimensional.

(6) X projective

$$\Rightarrow \quad H^i(X, \mathcal{F}) = 0 \quad \forall i > \dim(X)$$

(also true for X quasi projective)

Sheaf cohomology groups can be computed as the cohomology of the Čech cochain complex for an affine cover

$$X = \bigcup_{i \in I} U_i$$

$$\check{C}^n(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \cdots < i_n} \mathcal{F}|U_{i_0} \cap \cdots \cap U_{i_n})$$

with differential

$$d_n : \check{C}^n(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$$

$$\alpha = (\alpha_{j_0, \dots, j_n}) \longmapsto \left(\sum_{k=0}^{n+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{n+1}} | U_{i_0 \cdots i_{n+1}} \right)$$

$$\check{H}^n(\mathcal{U}, \mathcal{F}) := H^n(\check{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

<u>Theorem</u> X noetherian, separated $\mathcal{U} = (U_i)_{i \in I}$ affine open cover

$$\forall \mathcal{F} \in \mathrm{QCoh}(X) : \quad \check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F})$$

Cohomology of projective space

Some twists:

$$k[x_0,\dots,x_n](m) = \bigoplus_{s\in\mathbb{Z}} k[x_0,\dots,x_n]_{s-m}$$

graded module over $k[x_0,\dots,x_n]$

$\rightsquigarrow$ defines a quasicoherent sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$ which is a line bundle

$\cdot$ $\mathcal{O}_{\mathbb{P}^n}(m)|_{U_i} = x_i^{-m}\cdot\mathcal{O}_{U_i}$

with transition functions

$$\mathcal{O}(m)(U_i\cap U_j) \longrightarrow \mathcal{O}(m)(U_i\cap U_j)$$

$$f \longmapsto \frac{x_i^m}{x_j^m} f$$

Properties: 1) $\mathcal{O}(m)\otimes\mathcal{O}(n)\cong\mathcal{O}(m+n)$

2) $\mathcal{O}(1)$ generates $\mathrm{Pic}(\mathbb{P}^n)\cong\mathbb{Z}\,\mathcal{O}(1)$

Thm 1) $\exists$ iso of graded rings

$$k[x_0,\dots,x_n]\longrightarrow\bigoplus_{m\in\mathbb{Z}} H^0(\mathbb{P}^n,\mathcal{O}(m))$$

2) $H^n(\mathbb{P}^n,\mathcal{O}(-n-1))\cong k$

3) $H^i(\mathbb{P}^r,\mathcal{O}(m))=0 \quad 0<i<r \ \forall m$

4) $H^0(\mathbb{P}^r, \mathcal{O}(m)) \times H^r(\mathbb{P}^r, \mathcal{O}(-m-n-1))$
$\longrightarrow k$
is a perfect pairing.

<u>Pf</u> Take $\mathcal{U} = \{U_i\}_i$, $U_i = \{x_i \neq 0\}$

Čech cohomology computation:

$S = k[x_0, \dots, x_n]$

$S_{x_{i_0} \cdots x_{i_p}} = k[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}}$

$$\Gamma(U_i, \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m)) = \bigoplus_{m \in \mathbb{Z}} x_i^{-m} k[x_0, \dots, \hat{x}_i, \dots x_n]$$
$$= S_{x_i}$$

Compute $H^0(\mathcal{U}, \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m))$:

Begining of Čech complex

$$\bigoplus_{i=0}^{r} S_{x_i} \xrightarrow{d_0} \bigoplus_{i<j} S_{x_i x_j}$$

$$d_0(\alpha) = (\alpha_i - \alpha_j)_{i<j}$$

$d_0(\alpha) = 0 \iff \alpha_i - \alpha_j = 0 \quad \forall i<j$
$\Rightarrow \alpha_i = \alpha_j \in S_{x_i} \cap S_{x_j} \quad \forall i,j$

$\Leftrightarrow \alpha = \qquad \Rightarrow \alpha_i \in \cap$

$\Rightarrow H^0(\mathbb{P}^n, \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)) \cong S$

$\Rightarrow 1$

End of Čech complex

$$\bigoplus_{i=0}^{n} S_{x_0 \cdots \hat{x}_i \cdots x_n} \xrightarrow{d_n} S_{x_0 \cdots x_n}$$

$$\mathrm{im}(d_r) = \langle x_0^{l_0} \cdots x_n^{l_n} \mid \exists i \text{ s.t. } l_i \geq 0 \rangle$$

$$\Rightarrow H^n(\mathcal{U}, \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)) = \frac{S_{x_0 \cdots x_r}}{\mathrm{Im}(d_r)}$$

$$= \langle x_0^{l_1} \cdots x_r^{l_r} \mid \forall i \; l_i < 0 \rangle$$

$$= \bigoplus_{m\in\mathbb{Z}} \left\langle x_0^{l_1} \cdots x_r^{l_r} \;\middle|\; \begin{matrix}\forall i \; l_i < 0 \\ \sum l_i = m\end{matrix} \right\rangle$$

$\Rightarrow H^n(\mathcal{U}, \mathcal{O}(-n-1)) = k$

$H^n(\mathcal{U}, \mathcal{O}(m)) = 0 \quad$ for $m > -n-1$

$H^n(\mathcal{U}, \mathcal{O}(m)) \cong \binom{-m-1}{-n-m-1} \quad m \leq -n-1$

3) $\exists$ SES

$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$

$\rightsquigarrow$ SES

$$0 \to \mathcal{O}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_{\mathbb{P}^{n-1}}(m) \to 0$$

induction on n, m

LES $\Rightarrow$ $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$

for $0 < i < n-1$.

SES

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}(m-1)) \to H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(\mathbb{P}^{n-1}, \mathcal{O}(m-1)) \to 0$$

follows from Pascals rule $\binom{n+m}{m} = \binom{n+m-1}{m} + \binom{n+m-1}{m-1}$

similarly.

$$0 \to H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}(m)) \to H^n(\mathbb{P}^n, \mathcal{O}(m-1))$$
$$\to H^n(\mathbb{P}^n, \mathcal{O}(m)) \to 0$$

$\Rightarrow$ $H^i(\mathbb{P}^n, \mathcal{O}(m-1)) \xrightarrow{\cdot x_r} H^i(\mathbb{P}^n, \mathcal{O}(m))$ bijection.

but on the other hand. we know

Čech complex localized at $\cdot x_r = 0$ frocien

is exact (cohomology of affine)

$\Rightarrow$ $H^i(\mathbb{P}^n \mathcal{O}(m)) = 0$ as desired.

4) $H^0(\mathbb{P}^n, \mathcal{O}(m)) \times H^r(\mathbb{P}^n, \mathcal{O}(-m-n-1)) \to H^r(\mathcal{O}(-n-1))$

$\langle x_0^{a_0} \cdots x_n^{a_n} \mid \sum a_i = m, \ a_i \geq 0 \rangle \times \langle x_0^{b_0} \cdots x_n^{b_n} \mid \sum b_i = -m-n-1, \ b_i < 0 \rangle$

$\xrightarrow{\text{multiply}} \langle x_0^{c_0} \cdots x_n^{c_n} \mid \sum c_i = -n-1, \ c_i < 0 \rangle$

Canonical divisor

X smooth proj. / $k = \bar{k}$

Ω^1_X : sheaf of Kähler differentials
locally free sheaf or rank $\dim X$

$$\Omega^1_X(\operatorname{Spec}(A)) = \bigoplus_{a \in A} A\,da \Big/ \begin{array}{l} d(a+a') = da + da' \\ d(ab) = (da)b + a\,db \\ d(\lambda) = 0 \end{array}$$

$$\| $$

$$\operatorname{Spec}\left(\frac{k[x_1,\dots,x_n]}{(f_1,\dots,f_n)}\right) = \bigoplus_{i=1}^{n} A\,dx_i \Big/ \left(\frac{\partial f_j}{\partial x_i}\right)$$

$$\omega_X = \det(\Omega^1_X) = \Lambda^{\dim(X)} \Omega^1_X \qquad \text{canonical line bundle}$$

$$\omega_X \cong \mathcal{O}_X(K_X) \qquad K_X \text{ canonical divisor.}$$

Example: $X = \mathbb{P}^n$ Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

$$\Rightarrow \det(\Omega^1_{\mathbb{P}^n}) \otimes \det(\mathcal{O}) \cong \det(\mathcal{O}(-1)^{n+1})$$

$$\det(\Omega^1_{\mathbb{P}^n}) = \omega_{\mathbb{P}^n}, \qquad \det(\mathcal{O}(-1)^{n+1}) = \mathcal{O}(-1)^{\otimes n+1} = \mathcal{O}(-n-1)$$

$\Rightarrow$ have perfect pairing

$$H^i(\mathbb{P}^n, \mathcal{O}(m)) \times H^{n-i}(\mathbb{P}^n, \overset{\mathcal{O}(m)^\vee}{\overset{\shortparallel}{\mathcal{O}(-m)}} \otimes \omega_{\mathbb{P}^n})$$

i.e. $H^i(\mathbb{P}^n, \mathcal{O}(m)) \cong H^{n-i}(\mathbb{P}^n, \mathcal{O}(m)^\vee \otimes \omega_{\mathbb{P}^n})^\vee$

Theorem (Serre duality)

X smooth projective connected variety $k = \bar{k}$

$\mathcal{F}$ locally free sheaf.

$\exists$ perfect pairing

$$H^i(X, \mathcal{F}) \times H^{\dim(X)-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to k$$

$$\Rightarrow H^i(X, \mathcal{F}) \cong H^{\dim(X)-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*$$

Serre duality for curves:

X smooth proj. connected curve

$\mathcal{F}$ locally free sheaf

$\exists$ $H^0(X, \mathcal{L}) \times H^1(X, \mathcal{L}^\vee \otimes \Omega^1_X) \longrightarrow k$

perfect pairing.